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On the problem of the choice of first approximants in a two-sided iteration method

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Introduction

In the papers [2], [3] a two-sided iteration method is worked out for the general homogeneous boundary value and initial value problems of non-linear differential equations of order n under the assumption of the contractivity of the corresponding integral operator A. More precisely, two sequences of functions were constructed that approximated, together with their derivatives, from above and from below arbitrarily precisely and uniformly on the considered segment [0, 1] the solution of the boundary value problem and its derivatives. However, the construction of the approximants just mentioned depends on the assumption of the existence of first approximants z_1, w_1 that are in a well-defined relation with the second, and, which is even worse, with the third pair of approximants.

In this note we shall prove that under the only assumption of contractivity the first pair of approximants exists.

Let us consider the boundary value problem

$$(0.1)\qquad \begin{cases} y^{(n)}(x) = f[y] \equiv f(x, y(x), \ldots, y^{(n-1)}(x)) \quad (0 \leqq x \leqq 1, n \geqq 2), \\ L_i y = \sum_{k=0}^{n-1} \left(a_{ik} y^{(k)}(0) + b_{ik} y^{(k)}(1)\right) = 0 \quad (i = 0, \ldots, n-1) \end{cases}$$

with the given function

$$f(x, u_0, \ldots, u_{n-1}) : [0, 1] \times \mathscr{R} \times \ldots \times \mathscr{R} \to \mathscr{R}, \quad \left|\frac{\partial f}{\partial u_i}\right| \leqq N \quad (i = 0, \ldots, n-1)$$

that is continuous and continuously differentiable with respect to each u_i ($i=0, \ldots, n-1$), and with coefficients a_{ik}, b_{ik} such that the problem

$$y^{(n)}(x) = 0 \quad (0 \leqq x \leqq 1;\ n \geqq 2), \quad L_i y = 0 \quad (i = 0, \ldots, n-1)$$

has the only solution $y=0$.

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We remark that the general problem on $[a, b]$ with homogeneous boundary value condition leads to problem (0.1).

Problem (0.1) is equivalent to the integral equation

$$y = y(x) = Ay \equiv \int_0^1 G(x,t) f[y(t)]\,dt, \tag{0.2}$$

(where G is Green's function) in the space $\mathcal{M}$ of $n-1$ times continuously differentiable functions defined on $[0, 1]$ and satisfying the boundary value restriction. Let us introduce an ordering and a metric in $\mathcal{M}$ by the formulas

$$\begin{cases} z \leqq w \Leftrightarrow z^{(i)}(x) \leqq w^{(i)}(x) \quad (0 \leqq x \leqq 1;\ i = 0, \ldots, n-1), \\ \varrho(z, w) = \|z-w\| = \sum\limits_{i=0}^{n-1} \max\limits_{[0,1]} |z^{(i)}(x) - w^{(i)}(x)|. \end{cases} \tag{0.3}$$

Let us suppose that the condition

$$\mathrm{N} \sum_{i=0}^{n-1} \max_{[0,1]} \int_0^1 \left| \frac{\partial^i G(x,t)}{\partial x^i} \right| dt = \theta < 1 \tag{0.4}$$

is satisfied. This means that the operator A is strongly contractive in $\mathcal{M}$. In the paper [3] for problem (0.2) and for given $\varepsilon > 0$ we constructed an auxiliary function (minorant) $\tilde{G}(x, t)$, which is $n-1$ times differentiable with respect to x in each of the sectors

$$x_j \leqq x \leqq x_{j+1}, \quad 0 \leqq t \leqq x; \quad x_j \leqq x \leqq x_{j+1}, \quad x \leqq t \leqq 1 \quad (j = 0, \ldots, m-1),$$

where $\max\limits_{j=0,\ldots,m-1} (x_{j+1} - x_j)$ is sufficiently small (cf. the concluding part of the proof of the theorems),

$$x_0 = 0 < x_1 < \ldots < x_m = 1$$

are numbers, and along the straight lines $x = x_j$, $x = x_{j+1}$; $x = t$ the function $\tilde{G}$ or some of its derivatives with respect to x may be multivalued (they can have points of discontinuity of the first kind), moreover, the inequalities

$$N \sum_{i=0}^{n-1} \max_{[0,1]} \int_0^1 \left| \frac{\partial^i \tilde{G}(x,t)}{\partial x^i} \right| dt = \theta_1 \leqq \theta + \varepsilon < 1 \tag{0.5}$$

$$\frac{\partial^i \tilde{G}(x,t)}{\partial x^i} \leqq - \left| \frac{\partial^i G(x,t)}{\partial x^i} \right| \quad (i = 0, \ldots, n-1) \tag{0.6}$$

are satisfied.

Finally, we introduce the linear space $\tilde{\mathcal{M}}$ of $n-1$ times continuously differentiable functions defined on the segments $[x_j, x_{j+1}]$ $(j=0, \ldots, m-1)$. We consider

$\tilde{\mathcal{M}}$ with the following ordering and metric

$$(0.7)\qquad z \leqq w \Leftrightarrow \begin{cases} z^{(i)}(x) \leqq w^{(i)}(x) & \left(x \in \bigcup\limits_{j=0}^{m-1} (x_j, x_{j+1})\right), \\ z^{(i)}(x_j+0) \leqq w^{(i)}(x_j+0), \\ z^{(i)}(x_{j+1}-0) \leqq w^{(i)}(x_{j+1}-0) \\ (i = 0, \dots, n-1;\ j = 0, \dots, m-1), \end{cases}$$

$$(0.8)\qquad \varrho(z, w) = \|z-w\| = \sum_{i=0}^{n=1} \max_{[0,1]} |z^{(i)}(x) - w^{(i)}(x)|.$$

It is obvious that $\tilde{\mathcal{M}} \supset \mathcal{M}$ and that the ordering and metric of $\tilde{\mathcal{M}}$ are extensions of those of $\mathcal{M}$, thus our above notation is justified. It is also obvious that for the operator $\tilde{A}$ (the extension of A to $\tilde{\mathcal{M}}$), we have

$$(0.9)\qquad \tilde{A}z = \int_0^1 G(x,t) f[z(t)]\,dt = \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} G(x,t) f[z(t)]\,dt$$

and for any $z, w \in \tilde{\mathcal{M}}$

$$(0.10)\qquad \varrho(\tilde{A}z, \tilde{A}w) \leqq \theta \varrho(z, w).$$

Let us introduce the notation

$$\Delta_{z,w}(t) = \sum_{i=0}^{n-1} (z^{(i)}(t) - w^{(i)}(t)), \quad \tilde{B}(z, w) = N \int_0^1 \tilde{G}(x, t) \Delta_{z,w}(t)\,dt.$$

Let us define the operators $E, F \colon \tilde{\mathcal{M}} \times \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ in the following way

$$E(z, w) = \frac{1}{2}(\tilde{A}z + \tilde{A}w) + \frac{1}{2}\tilde{B}(z, w),$$

$$F(z, w) = \frac{1}{2}(\tilde{A}z + \tilde{A}w) - \frac{1}{2}\tilde{B}(z, w).$$

Consider the iteration process

$$(0)\qquad z_{p+1} = E(z_p, w_p), \quad w_{p+1} = F(z_p, w_p) \quad (p = 1, 2, \dots).$$

We remark that E is non-increasing in z and non-decreasing in w; F is non-decreasing in z and non-increasing in w.

In the so-called monotone case of problem (0.1), i.e., when

$$\frac{\partial^i G(x,t)}{\partial x^i} \leqq 0 \quad (0 \leqq x, t \leqq 1; \quad i = 0, \dots, n-1),$$

or when the derivatives of G are non-negative, $\tilde{G}$ can be taken to be equal to $-|G|$ and $\tilde{\mathcal{M}}$ can be taken to be equal to $\mathcal{M}$ and one can consider in it the approximation according to the rula (0).

I.

Let us consider first the case of problem (0.1) when the following strong assumption of contractivity is satisfied:

$$\text{(i)} \qquad N\sum_{i=0}^{n-1}\max_{[0,1]}\int_0^1\left|\frac{\partial^i G(x,t)}{\partial x^i}\right|dt=\tilde{\theta}<\frac{1}{2}.$$

It is obvious that in this case there exists a minorant with the required properties of smoothness and for which we have

$$\text{(1.1)} \qquad N\sum_{i=0}^{n-1}\max_{[0,1]}\int_0^1\left|\frac{\partial^i \tilde{G}(x,t)}{\partial x^i}\right|dt=\tilde{\tilde{\theta}}<\frac{1}{2}.$$

Lemma 1.1. *Under the assumption of* (i) *there exist* $z_1, w_1 \in \tilde{\mathcal{M}}$ *such that*

$$\text{(1.2)} \qquad z_2 \leqq z_1, \quad w_1 \leqq w_2.$$

Proof. Let us take a positive element φ from $\tilde{\mathcal{M}}$, i.e., a function $\varphi(x)$ for which

$$\varphi^{(i)}(x)>0 \quad (i=0,\dots,n-1;\ 0\leqq x\leqq 1).$$

The assertion of our lemma follows from the fact that by virtue of (1.1) the system of equations

$$\text{(1.3)} \qquad E(z_1,w_1)=z_1-\varphi, \quad F(z_1,w_1)=w_1+\varphi$$

has a solution in $\tilde{\mathcal{M}}\times\tilde{\mathcal{M}}$.

Theorem 1.1. *Under the assumption of* (i) *there exist* $z_1, w_1 \in \tilde{\mathcal{M}}$ *for which*

$$z_2\leqq z_1, \quad z_3\leqq z_1; \quad w_1\leqq w_2, \quad w_1\leqq w_3.$$

Proof. The elements $z_1, w_1 \in \tilde{\mathcal{M}}$ can be taken as in Lemma 1.1. On account of the monotonicity of E, F we then have

$$z_2\leqq z_1, \quad z_2\leqq z_3; \quad w_1\leqq w_2, \quad w_3\leqq w_2.$$

It remains to prove that the function φ can be chosen so that the inequalities

$$\text{(1.4)} \qquad z_3\leqq z_1, \quad w_1\leqq w_3$$

are also satisfied. These inequalities are equivalent to the inequalities

$$\text{(1.5)} \qquad \begin{cases} 0\leqq \varphi+[z_1-\varphi-E(z_1-\varphi,w_1+\varphi)] \\ -\varphi+[w_1+\varphi-F(z_1-\varphi,w_1+\varphi)]\leqq 0. \end{cases}$$

We remark that by Lagrange's formula

$$f[z_1-\varphi]-f[z_1] = -\sum_{i=0}^{n-1} \frac{\partial f}{\partial u_i}\bigg|_{P(x)} \varphi^{(i)},$$

$$f[w_1+\varphi]-f[w_1] = \sum_{i=0}^{n-1} \frac{\partial f}{\partial u_i}\bigg|_{Q(x)} \varphi^{(i)},$$

where we denoted by $\left.\frac{\partial f}{\partial u_i}\right|_{P(x)}, \left.\frac{\partial f}{\partial u_i}\right|_{Q(x)}$ the values of these derivatives at the corresponding points of the $n+1$ dimensional space (in the sequel the letters P, Q shall be often omitted). By using these formulas it is easy to prove that the inequalities (1.5), understood in the sense of the ordering of $\tilde{\mathscr{M}}$, are equivalent to the inequalities

$$\varphi^{(r)}(x) \geqq \frac{1}{2}\int_0^1 \sum_{i=0}^{n-1}\left[\pm\frac{\partial^r G(x,t)}{\partial x^r}\left(\left.\frac{\partial f}{\partial u_i}\right|_P - \left.\frac{\partial f}{\partial u_i}\right|_Q\right) - 2N\frac{\partial^r \tilde{G}(x,t)}{\partial x^r}\right]\varphi^{(i)}(t)\,dt \tag{1.6}$$

$$(r = 0, \ldots, n-1; \quad 0 \leqq x \leqq 1).$$

Let us rewrite inequality (1.1) in the form

$$\sum_{i=0}^{n-1} \max_{0\leqq x\leqq 1} \int_0^1 -2N\frac{\partial^i \tilde{G}(x,t)}{\partial x^i}\,dt = 2\tilde{\tilde{\theta}} = \theta^* < 1. \tag{1.7}$$

Let us denote by $\theta^*\lambda_i$ the i-th term in this sum. We have

$$\lambda_0, \ldots, \lambda_{n-1} > 0; \quad \sum_{i=0}^{n-1} \lambda_i = 1 \tag{1.8}$$

(cf. the construction of $\tilde{G}$ in [3]). It is easy to verify that the function

$$\varphi(x) = \begin{cases} \delta \sum_{i=0}^{n-1} \frac{\lambda_i}{i!} x^i & (0 \leqq x \leqq a), \\ \delta \sum_{i=0}^{n-1} \frac{\lambda_i}{i!}(x-a_j)^i & (a_j = j\cdot a \leqq x \leqq a_{j+1} = (j+1)a), \end{cases}$$

where $\delta > 0$ is an arbitrary constant, $a_0=0$; $j=0, \ldots, k$; $(k+1)a=1$, for sufficiently small $a>0$ satisfies inequality (1.6). This follows from (1.7), (1.8) and from the fact that for small $a>0$ the ratio of the maximum and minimum of $\varphi^{(i)}$ is close to the unit for every $i=0, \ldots, n-1$.

Remark 1.1. Since the change to a finer division of the segment [0, 1] (cf. [3]) in the construction of $\tilde{G}$ and $\tilde{\mathscr{M}}$ does not bother those properties of $\tilde{G}$ and $\tilde{\mathscr{M}}$ that are needed by us, the initial division

$$x_0 = 0 < x_1 < \ldots < x_m = 1$$

of the segment [0, 1] can be chosen to be equal to

$$x_0 = 0, \quad x_1 = a, \ldots, x_m = (k+1)a = 1 \quad (k+1 = m),$$

where a denotes the number in the proof of Theorem 1.1.

Remark 1.2. To our regret we did not succeed in proving the existence of the first pair (z_1, w_1) from $\mathcal{M}\times\mathcal{M}$ in an analogous way as in Theorem 1.1, since the ratios of the maximums and minimums corresponding to the function $\varphi\in\mathcal{M}$ cannot be made arbitrarily close to the unit.

Remark 1.3. We note that instead of the accurate solution of system (1.3) one may take its approximate solution $(z^{(1)}, w^{(1)})$ obtained, for example, by the method of successive approximation starting with an arbitary pair $(z, w)\in\tilde{\mathcal{M}}\times\tilde{\mathcal{M}}$. Hence from the practical point of view Theorem 1.1. may be useful. An analogous statement concerns all the following theorems of this paper.

II.

In connection with the negative statement in Remark 1.2 in this section we shall elaborate another method of the construction of the pair (z_1, w_1) from $\mathcal{M}\times\mathcal{M}$. This method relies on the usage of defect functions (cf. [1], [2], [3]). Actually, we shall consider only the monotone case of problem (0.1), i.e., when

$$\frac{\partial^i G(x,t)}{\partial x^i} \leqq 0 \quad (i = 0, \ldots, n-1;\ 0 \leqq x, t \leqq 1)$$

besides, we suppose that at all points

$$\frac{\partial f}{\partial u_i} \geqq 0 \quad (i = 0, \ldots, n-1).$$

The case when all partial derivatives of f are non-positive is entirely obvious (cf. [2]).

In this case the approximants z_p, w_p may be constructed independently of each other according to the rule

$$\text{(A)} \qquad z_{p+1} = Az_p, \quad w_{p+1} = Aw_p \quad (p = 1, 2, \ldots).$$

Theorem 2.1. *Under the weak assumption of contractivity* (0.4) *there exist elements $z_1, w_1\in\mathcal{M}$ for which in the sense of the ordering of $\mathcal{M}$ we have*

$$\text{(2.1)} \qquad z_2 \leqq z_1, \quad z_3 \leqq z_1; \quad w_1 \leqq w_2, \quad w_1 \leqq w_3.$$

Proof. We remark that the rule (A) is equivalent to the rule

$$(\mathrm{A}') \qquad \begin{cases} z_{p+1}(x) = z_p(x)-\sigma_p(x), \quad w_{p+1}(x) = w_p(x)-\eta_p(x), \\ \sigma_p(x) = \int\limits_0^1 G(x,t)\alpha_p(t)\,dt, \quad \eta_p(x) = \int\limits_0^1 G(x,t)\beta_p(t)\,dt, \\ \alpha_p(x) = z_p^{(n)}(x)-f[z_p], \quad \beta_p(x) = w_p^{(n)}(x)-f[w_p] \\ (0 \leqq x \leqq 1;\ p = 1, 2, \ldots). \end{cases}$$

Consequently, we shall seek $z_1, w_1 \in \mathcal{M}$ for which

(2.2)
$$\alpha_1(x) < 0, \quad \alpha_1(x)+\alpha_2(x) \leqq 0; \quad \beta_1(x) > 0, \quad \beta_1(x)+\beta_2(x) \geqq 0 \quad (0 \leqq x \leqq 1).$$

To attain the negativeness of α_1 it is enough to take the solution $z_1 \in \mathcal{M}$ of the equation

$$z^{(n)}(x)-f[z(x)] = -c = \alpha_1.$$

This equation is solvable in $\mathcal{M}$ by virtue of the contractivity assumption (0.4). For this function z_1 the second inequality in (2.2) is also satisfied as because of the rule (A′) we have

$$\alpha_1(x)+\alpha_2(x) = z_1^{(n)}(x)-f[z_1]+z_1^{(n)}(x)-\sigma_1^{(n)}(x)-f[z_2].$$

Thus by using Lagrange's formula for $f[z_1]-f[z_2]$ we arrive at the inequality

$$\alpha_1(x)+\alpha_2(x) = \alpha_1(x)+\sum_{i=0}^{n-1} \left.\frac{\partial f}{\partial u_i}\right| \sigma_1^{(i)}(x) \leqq 0.$$

This inequality is equivalent to the inequality

$$1 \geqq \sum_{i=0}^{n-1} \left|\frac{\partial f}{\partial u_i}\right| \int\limits_0^1 \left|\frac{\partial^i G(x,t)}{\partial x^i}\right| dt$$

the validity of which follows from the contractivity condition (0.4).

The existence of w_1 may be proved in a similar way.

Remark 2.1. In the case when

$$\frac{\partial^i G(x,t)}{\partial x^i} \geqq 0, \quad \frac{\partial f}{\partial u_i} \leqq 0 \quad (i = 0, \ldots, n-1;\ 0 \leqq x, t \leqq 1)$$

we may also prove the existence of $z_1, w_1 \in \mathcal{M}$.

III.

Under the strong assumption of contractivity (i) let us now consider the monotone case of problem (0.1), i.e.,

$$\frac{\partial^i G(x,t)}{\partial x^i} \leqq 0 \quad (0 \leqq x, t \leqq 1;\ i = 0, \dots, n-1)$$

however, the derivatives of f are not all of the same sign, and themselves may change sign, too. The approximants z_p, w_p shall now be constructed in $\mathcal{M}$ according to the rule (0), i.e., we have

$$\text{(B)} \quad \begin{cases} z_{p+1} = E(z_p, w_p) = \dfrac{1}{2} \displaystyle\int_0^1 G(x,t)\big(f[z_p(t)] + f[w_p(t)]\big)dt + \\ \qquad + \dfrac{N}{2} \displaystyle\int_0^1 G(x,t) \sum_{i=0}^{n-1} \big(z_p^{(i)}(t) - w_p^{(i)}(t)\big)dt, \\ w_{p+1} = F(z_p, w_p) = \dfrac{1}{2} \displaystyle\int_0^1 G(x,t)\big(f[z_p(t)] + f[w_p(t)]\big)dt - \\ \qquad - \dfrac{N}{2} \displaystyle\int_0^1 G(x,t) \sum_{i=0}^{n-1} \big(z_p^{(i)}(t) - w_p^{(i)}(t)\big)dt. \end{cases}$$

Theorem 3.1. *Under the assumption of* (i) *in the case considered there exist elements* $z_1, w_1 \in \mathcal{M}$ *for which*

$$\text{(3.1)} \qquad z_2 \leqq z_1, \quad z_3 \leqq z_1; \quad w_1 \leqq w_2, \quad w_1 \leqq w_3$$

in the sense of the ordering of $\mathcal{M}$.

Proof. We note that rule (B) is equivalent to the rule

$$\text{(B')} \quad \begin{cases} z_{p+1}(x) = z_p(x) - \sigma_p(x), \quad w_{p+1}(x) = w_p(x) - \eta_p(x), \\ \sigma_p(x) = \displaystyle\int_0^1 G(x,t)\alpha_p(t)dt, \quad \eta_p(x) = \displaystyle\int_0^1 G(x,t)\beta_p(t)dt, \\ \alpha_p(x) = z_p^{(n)}(x) - \dfrac{1}{2} f[z_p] - \dfrac{1}{2} f[w_p] - \dfrac{N}{2} \displaystyle\sum_{i=0}^{n-1} \big(z_p^{(i)}(x) - w_p^{(i)}(x)\big), \\ \beta_p(x) = w_p^{(n)}(x) - \dfrac{1}{2} f[z_p] - \dfrac{1}{2} f[w_p] + \dfrac{N}{2} \displaystyle\sum_{i=0}^{n-1} \big(z_p^{(i)}(x) - w_p^{(i)}(x)\big) \\ (0 \leqq x \leqq 1;\ p = 1, 2, \dots). \end{cases}$$

On the other hand, the inequalities (3.1) are equivalent to the inequalities

$$\text{(3.2)} \quad \begin{cases} \alpha_1(x) \leqq 0, \quad \alpha_1(x) + \alpha_2(x) \leqq 0; \quad \beta_1(x) \geqq 0, \quad \beta_1(x) + \beta_2(x) \geqq 0 \\ (0 \leqq x \leqq 1). \end{cases}$$

The solution $(z_1, w_1)\in\mathcal{M}\times\mathcal{M}$ of the system

$$
(3.3)\qquad
\begin{cases}
z^{(n)}(x)-\dfrac{1}{2}f[z]-\dfrac{1}{2}f[w]-\dfrac{N}{2}\sum\limits_{i=0}^{n-1}\left(z^{(i)}(x)-w^{(i)}(x)\right)=-c,\\[2mm]
w^{(n)}(x)-\dfrac{1}{2}f[z]-\dfrac{1}{2}f[w]+\dfrac{N}{2}\sum\limits_{i=0}^{n-1}\left(z^{(i)}(x)-w^{(i)}(x)\right)=c,
\end{cases}
$$

where $c>0$ is a constant, exists by virtue of (i) and obviously satisfies the first and the third inequalities under (3.2). Moreover, z_1, w_1 also satisfies the second and the fourth inequalities in (3.2). Indeed, by the rule (B′) and by using Lagrange's formula for the difference

$$f[z_1]-f[z_1-\sigma_1],\quad f[w_1]-f[w_1-\eta_1]$$

one can show that, for example, the second inequality in (3.2) is equivalent to the inequality

$$\alpha_1(x)=-c\leqq\frac{1}{2}\sum_{i=0}^{n-1}\left\{\sigma_1^{(i)}(x)\left[\left|\frac{\partial f}{\partial u_i}\right|+N\right]+\eta_1^{(i)}(x)\left[\left|\frac{\partial f}{\partial u_i}\right|-N\right]\right\}.$$

If we replace $\sigma_1^{(i)}$ and $\eta_1^{(i)}$ by their expressions via Green's function and α_1, β_1, we arrive at the inequality

$$
(3.4)\qquad 1\geqq\sum_{i=0}^{n-1}\frac{1}{2}\left[\left(\left.\frac{\partial f}{\partial u_i}\right|_P-\left.\frac{\partial f}{\partial u_i}\right|_Q\right)+2N\right]\int_0^1\left|\frac{\partial^i G(x,t)}{\partial x^i}\right|dt.
$$

The validity of (3.4) follows from condition (i) and from the condition

$$\left|\frac{\partial f}{\partial u_i}\right|\leqq N\quad(i=0,\dots,n-1).$$

The fourth inequality under (3.2) may be proved in an analogous way.

Remark 3.1. The case

$$\frac{\partial^i G(x,t)}{\partial x^i}\geqq 0\quad(i=0,\dots,n-1;\quad 0\leqq x,t\leqq 1)$$

may be handled analogously.

IV.

Now we shall give still another method, concerning the general G and rule (0), for the construction of functions from $\tilde{\mathcal{M}}$ that satisfy the inequalities (1.2). This method shall be very useful later.

Let us take two arbitrary functions $(z_1, w_1)\in\tilde{\mathcal{M}}\times\tilde{\mathcal{M}}$. Let the operators E and F map (z_1, w_1) into (z_2, w_2). We shall seek the elements $Z_1, W_1\in\tilde{\mathcal{M}}$ that satisfy the inequalities (1.2) in the form

$$Z_1 = z_1+\varphi_1, \quad W_1 = w_1+\eta_1. \tag{4.1}$$

In this case

$$Z_2(x) =$$
$$= \frac{1}{2}\int_0^1 \left\{G(x,t)\left(f[z_1+\varphi_1]+f[w_1+\eta_1]\right)+\tilde{G}(x,t)N\sum_{i=0}^{n-1}\left((z_1+\varphi_1)^{(i)}-(w_1+\eta_1)^{(i)}\right)\right\}dt,$$
$$W_2(x) =$$
$$= \frac{1}{2}\int_0^1 \left\{G(x,t)\left(f[z_1+\varphi_1]+f[w_1+\eta_1]\right)-\tilde{G}(x,t)N\sum_{i=0}^{n-1}\left((z_1+\varphi_1)^{(i)}-(w_1+\eta_1)^{(i)}\right)\right\}dt.$$

By Lagrange's formula we have

$$f[z_1+\varphi_1] = f[z_1]+\sum_{i=0}^{n-1}\varphi_1^{(i)}\left.\frac{\partial f}{\partial u_i}\right|_P, \quad f[w_1+\eta_1] = f[w_1]+\sum_{i=0}^{n-1}\eta_1^{(i)}\left.\frac{\partial f}{\partial u_i}\right|_Q.$$

Consequently, we have to satisfy the following inequalities in $\tilde{\mathcal{M}}$:

(4.2)

$$\begin{cases} \frac{1}{2}\int_0^1\sum_{i=0}^{n-1}\left\{\varphi_1^{(i)}(t)\left[G(x,t)\left.\frac{\partial f}{\partial u_i}\right|_P+\tilde{G}(x,t)N\right]+\eta_1^{(i)}(t)\left[G(x,t)\left.\frac{\partial f}{\partial u_i}\right|_Q-\tilde{G}(x,t)N\right]\right\}dt \leqq \\ \qquad \leqq \varphi_1(x)+z_1(x)-z_2(x), \\ \frac{1}{2}\int_0^1\sum_{i=0}^{n-1}\left\{\varphi_1^{(i)}(t)\left[G(x,t)\left.\frac{\partial f}{\partial u_i}\right|_P-\tilde{G}(x,t)N\right]+\eta_1^{(i)}(t)\left[G(x,t)\left.\frac{\partial f}{\partial u_i}\right|_Q+\tilde{G}(x,t)N\right]\right\}dt \geqq \\ \qquad \geqq \eta_1(x)+w_1(x)-w_2(x). \end{cases}$$

For the given z_1, z_2, w_1, w_2 we obviously can choose φ_1, η_1 from $\tilde{\mathcal{M}}$ such that the inequalities

(4.3)

$$\left.\begin{cases} \varphi_1(x) \geqq |z_1(x)-z_2(x)|, \ldots, \varphi_1^{(n-1)}(x) \geqq |z_1^{(n-1)}(x)-z_2^{(n-1)}(x)|, \\ \eta_1(x) \leqq -|w_1(x)-w_2(x)|, \ldots, \eta_1^{(n-1)}(x) \leqq -|w_1^{(n-1)}(x)-w_2^{(n-1)}(x)| \end{cases}\right\} \quad (0 \leqq x \leqq 1)$$

are satisfied. Hence the left and right sides (and then their derivatives, too) of the inequalities (4.2) either coincide or have different signs.

Thus we have proved the following lemma.

Lemma 4.1. *For any elements z_1, w_1 of $\tilde{\mathcal{M}}$ the functions*

$$Z_1(x) = z_1(x)+\varphi_1(x), \quad W_1(x) = w_1(x)+\eta_1(x)$$

together with the auxiliary functions φ_1, η_1 *satisfying* (4.3) *also satisfy the inequalities* (1.2):

$$Z_2 = E(Z_1, W_1) \leqq Z_1, \quad W_1 \leqq W_2 = F(Z_1, W_1).$$

Consequence 4.1. Because of what we said in the concluding part of the introduction, Lemma 4.1 gives a method of construction for z_1, w_1; φ_1, η_1 $(\in \mathcal{M})$ such that with the functions $Z_1 = z_1 + \varphi_1$, $W_1 = w_1 + \eta_1$ we have the inequalities:

$$Z_2 = E(Z_1, W_1) \leqq Z_1, \quad W_1 \leqq W_2 = F(Z_1, W_1).$$

In this case for the given z_1, w_1 we have to choose $\varphi_1 > 0$, $\eta_1 < 0$ from $\mathcal{M}$ in such a way that

$$\text{(4.3')} \qquad \varphi_1 \geqq z_1 - z_2,\ z_2 - z_1; \quad \eta_1 \leqq w_1 - w_2,\ w_2 - w_1$$

is satisfied in the sense of the ordering of $\mathcal{M}$. Let us take two numbers $K < 0$, $L > 0$ for which

$$-K \geqq \max_{0 \leqq x \leqq 1} |z_1^{(n)}(x) - x_2^{(n)}(x)|, \quad L \geqq \max_{0 \leqq x \leqq 1} |w_1^{(n)}(x) - w_2^{(n)}(x)|.$$

Then the functions

$$\varphi_1(x) = \int_0^1 KG(x, t)\, dt, \quad \eta_1(x) = \int_0^1 LG(x, t)\, dt$$

satisfy (4.3').

Now we shall prove the existence of the first approximating pair $(z_1, w_1) \in \mathcal{M} \times \mathcal{M}$ in the monotone case of problem (0.1) under the weak assumption of contractivity (0.4), i.e., when

$$N \sum_{i=0}^{n-1} \max_{0 \leqq x \leqq 1} \int_0^1 \left| \frac{\partial^i G(x, t)}{\partial x^i} \right| dt = \theta < 1,$$

where $N > 0$ majorizes the modules of the partial derivatives of f.

Let us take an arbitrary negative number d and consider the problem

$$\text{(4.4)} \qquad \varphi_1^{(n)}(x) - N \sum_{i=0}^{n-1} \varphi_1^{(i)}(x) = d, \quad \varphi_1 \in \mathcal{M}.$$

Lemma 4.2. *The solution of problem* (4.4) *exists and is unique. It satisfies the inequality*

$$\text{(4.5)} \qquad \varphi_1^{(n)}(x) \leqq d(1 - \theta) \quad (0 \leqq x \leqq 1).$$

Proof. The existence and uniqueness of the solution follows from the contractivity condition (0.4). For the proof of (4.5) let us consider the problem

$$\text{(4.6)} \qquad u^{(n)}(x) = N \sum_{i=0}^{n-1} u^{(i)}(x), \quad u \in \mathcal{M}.$$

This problem has the only trivial solution $u=0$ in virtue of (0.4). With respect to the equation (4.6) the solution φ_1 of problem (4.4) and the corresponding φ_2 have defect functions (cf. for example [1], section 4.)

$$\tilde{\alpha}_1(x) = d, \quad \tilde{\alpha}_2(x) = dN \sum_{i=0}^{n-1} \int_0^1 \frac{\partial^i G(x,t)}{\partial x^i} dt.$$

Consequently, (0.4) and Theorem 4.1 of [1] imply that

$$\left.\begin{aligned} \varphi_1^{(n)}(x) - \varphi_3^{(n)}(x) = \tilde{\alpha}_1(x) + \tilde{\alpha}_2(x) \leqq d(1-\theta), \\ \varphi_1^{(n)}(x) \leqq \varphi_3^{(n)}(x) \leqq 0 \end{aligned}\right\} \quad (0 \leqq x \leqq 1).$$

Hence we have

$$\varphi_1^{(n)}(x) \leqq d(1-\theta) \quad (0 \leqq x \leqq 1).$$

Theorem 4.1. *In the monotone case of problem* (0.1) *under the contractivity assumption* (0.4) *there exist elements* $Z_1, W_1 \in \mathcal{M}$ *such that*

$$Z_2 = E(Z_1, W_1) \leqq Z_1, \quad W_1 \leqq W_2 = F(Z_1, W_1), \tag{4.7}$$

$$Z_3 = E(Z_2, W_2) \leqq Z_1, \quad W_1 \leqq W_3 = F(Z_2, W_2). \tag{4.8}$$

Proof. Let us take an arbitrary positive number $\varepsilon > 0$ and two functions z_1, w_1 from $\mathcal{M}$ so close to the solution y of problem (0.1) (z_1 and w_1 can be constructed by means of successive approximation) as to satisfy the inequalities

$$|z_1^{(n)}(x) - z_2^{(n)}(x)| < \varepsilon, \quad |w_1^{(n)}(x) - w_2^{(n)}(x)| < \varepsilon \quad (0 \leqq x \leqq 1).$$

Let us now seek Z_1, W_1 as in Lemma 4.1. in the form

$$Z_1 = z_1 + \varphi_1, \quad W_1 = w_1 + \eta_1$$

with unknown functions $\varphi_1, -\eta_1 \geqq 0$ from $\mathcal{M}$ (in the sense of the ordering of $\mathcal{M}$). In order that (4.7) be satisfied, φ_1 and η_1 have to satisfy (4.3′) and for such φ_1, η_1 the fulfilment of (4.8) is equivalent to the assertion that the positive functions

$$\hat{\varphi} = Z_1 - Z_2, \quad \hat{\eta} = W_2 - W_1$$

occurring in Lemma 4.1 satisfy the inequalities (1.6). These inequalities (in this case $\tilde{G} = G$, $\tilde{\mathcal{M}} = \mathcal{M}$) are equivalent to the inequalities

$$\begin{aligned} \hat{\alpha}_1(x) \leqq \frac{1}{2} \sum_{i=0}^{n-1} \Bigg[\left(\frac{\partial f}{\partial u_i}\Big|_P + N \right) \int_0^1 - \frac{\partial^i G(x,t)}{\partial x^i} \hat{\alpha}_1(t)\,dt + \\ + \left(\frac{\partial f}{\partial u_i}\Big|_Q - N \right) \int_0^1 - \frac{\partial^i G(x,t)}{\partial x^i} \hat{\beta}_1(t)\,dt \Bigg], \\ \hat{\beta}_1(x) \geqq \frac{1}{2} \sum_{i=0}^{n-1} \Bigg[\left(\frac{\partial f}{\partial u_i}\Big|_P - N \right) \int_0^1 - \frac{\partial^i G(x,t)}{\partial x^i} \hat{\alpha}_1(t)\,dt + \\ + \left(\frac{\partial f}{\partial u_i}\Big|_Q + N \right) \int_0^1 - \frac{\partial^i G(x,t)}{\partial x^i} \hat{\beta}_1(t)\,dt \Bigg], \end{aligned} \tag{4.9}$$

where we have

$$\hat{\alpha}_1 = Z_1^{(n)} - Z_2^{(n)} \leqq 0, \quad \hat{\beta}_1 = W_1^{(n)} - W_2^{(n)} \geqq 0$$

as because of (4.3′) $Z_1 \geqq Z_2$ and $W_1 \leqq W_2$.

The derivatives $\left.\frac{\partial f}{\partial u_i}\right|_P$, $\left.\frac{\partial f}{\partial u_i}\right|_Q$ in (4.9) are taken from the formulas

$$f[z_1]-f[z_2] = \sum_{i=0}^{n-1} \left.\frac{\partial f}{\partial u_i}\right|_P (z_1-z_2)^{(i)},$$

$$f[w_1]-f[w_2] = \sum_{i=0}^{n-1} \left.\frac{\partial f}{\partial u_i}\right|_Q (w_1-w_2)^{(i)}.$$

By using the formulas of Lemma 4.1 and taking $\varphi_1 = -\eta_1 > 0$ for the sake of simplicity we obtain

(4.10)

$$\hat{\alpha}_1(x) = z_1^{(n)}(x) - z_2^{(n)}(x) + \left(\varphi_1^{(n)}(x) - N \sum_{i=0}^{n-1} \varphi_1^{(i)}(x)\right) - \frac{1}{2} \sum_{i=0}^{n-1} \varphi_1^{(i)}(x) \left(\left.\frac{\partial f}{\partial u_i}\right|_P - \left.\frac{\partial f}{\partial u_i}\right|_Q\right),$$

$$\hat{\beta}_1(x) = w_1^{(n)}(x) - w_2^{(n)}(x) - \left(\varphi_1^{(n)}(x) - N \sum_{i=0}^{n-1} \varphi_1^{(i)}(x)\right) - \frac{1}{2} \sum_{i=0}^{n-1} \varphi_1^{(i)}(x) \left(\left.\frac{\partial f}{\partial u_i}\right|_P - \left.\frac{\partial f}{\partial u_i}\right|_Q\right).$$

In Theorem 3.1 everything went well essentially because we succeeded in finding z_1, w_1 such that the defect functions α_1, β_1 turned out to be constants ($-c$ resp. c), hence the necessary properties of z_1, w_1 immediately followed from the contractivity assumption.

The circumstances are similar now. For the $\varepsilon > 0$, z_1, w_1 already chosen we choose a number $d < 0$ in such a way that with the solution φ_1 of problem (4.4) the corresponding inequalities (4.3′) be satisfied. To achieve this it is enough (cf. Consequence 4.1) to take $d = -\sqrt{\varepsilon}(1-\theta)^{-1}$ (in the sequel we suppose that $0 < \varepsilon < 1$) as because of (4.5) the inequality

$$\varphi_1^{(n)}(x) \leqq d(1-\theta) \leqq -\varepsilon < -\max_{0 \leqq x \leqq 1} |z_1^{(n)}(x) - z_2^{(n)}(x)|$$

implies inequality (4.3′) for φ_1. One can prove analogously (taking $\eta_1 = -\varphi_1$ for the sake of simplicity) that for $\eta_1 = -\varphi_1$ the corresponding inequalities (4.3′) are also satisfied. The inequalities (4.7) are also satisfied for any $0 < \varepsilon < 1$.

In contrast with Section III now we may use the continuity of the partial derivatives of f, i.e., that

$$h(\varepsilon) = \max_{i,x} \left| \left.\frac{\partial f}{\partial u_i}\right|_{P(x)} - \left.\frac{\partial f}{\partial u_i}\right|_{Q(x)} \right| \to 0 \quad (\varepsilon \to 0).$$

From equality (4.4) it is easy to obtain that

$$\sum_{i=0}^{n-1} \varphi_1^{(i)}(x) \leqq \frac{-d\theta}{N} \quad (0 \leqq x \leqq 1),$$

whence in (4.10) we get that

$$\max_{0\leq x\leq 1}\hat{\alpha}_1(x) \geqq -\varepsilon-\frac{\sqrt{\varepsilon}}{1-\theta}-\frac{1}{2}|h(\varepsilon)|\frac{\theta\sqrt{\varepsilon}}{N(1-\theta)} = -\sqrt{\varepsilon}\left(\frac{1}{1-\theta}+o(\varepsilon)\right)-\varepsilon,$$

$$\max_{0\leq x\leq 1}\hat{\alpha}_1(x) \leqq -\sqrt{\varepsilon}\left(\frac{1}{1-\theta}+o(\varepsilon)\right)+\varepsilon$$

and analogously

$$\sqrt{\varepsilon}\left(\frac{1}{1-\theta}+o(\varepsilon)\right)-\varepsilon \leqq \hat{\beta}_1 \leqq \sqrt{\varepsilon}\left(\frac{1}{1-\theta}+o(\varepsilon)\right)+\varepsilon.$$

i.e., although $\hat{\alpha}_1, \hat{\beta}_1$ are not constant, the ratio of the maximum and minimum of each of $\hat{\alpha}_1, \hat{\beta}_1$ tends to $+1$ as $\varepsilon\to 0$ and the ratio of the maxima of $\hat{\alpha}_1$ and $\hat{\beta}_1$ tends to -1 as $\varepsilon\to 0$. Consequently, there exists $\varepsilon>0$ for which (4.9) is satisfied. To complete the proof one has to use condition (0.4), and the continuity of the partial derivatives of f.

V.

Finally let us consider the general G under the assumption of (0.4). The space $\tilde{\mathcal{M}}$, the functions $\tilde{G}$, $\tilde{\varphi}$ will be assumed to correspond to a sufficiently fine division of $[0,1]$. Let us take the function $\tilde{\varphi}=\varphi$ from Section I with undetermined $\delta>0$. Besides, in this case

$$\sum_{i=0}^{n-1}\max_{0\leq x\leq 1}\int_0^1 -N\frac{\partial^i\tilde{G}(x,t)}{\partial x^i}\,dt \equiv \sum_{i=0}^{n-1}\lambda_i\theta_1 = \theta_1 < 1.$$

Let us consider the problem

$$\varphi_1(x)-A_0\varphi_1 \equiv \varphi_1(x)-N\int_0^1\tilde{G}(x,t)\left(\sum_{i=0}^{n-1}\varphi_1^{(i)}(t)\right)dt = \tilde{\varphi}(x), \quad \varphi_1\in\tilde{\mathcal{M}}. \tag{5.1}$$

Lemma 5.1. *The solution φ_1 of problem* (5.1) *exists, is unique, and satisfies the inequality (in the sense of the ordering of $\tilde{\mathcal{M}}$)*

$$\varphi_1 \geqq c\tilde{\varphi} \tag{5.2}$$

with a general constant $c>0$ for all fine enough divisions of the segment $[0, 1]$.

Proof. The existence and uniqueness of the solution φ_1 of problem (5.1) follow from (0.4). To prove (5.2) let us consider the problem

$$u(x)-A_0u = 0, \quad u\in\tilde{\mathcal{M}}, \tag{5.3}$$

which in virtue of (0.4) has the only solution $u=0$. With respect to equation (5.3) for φ_1 we have the following inequality in $\tilde{\mathcal{M}}$:

$$\varphi_1 = \tilde{\varphi}+A_0\varphi_1 \geqq A_0\varphi_1. \tag{5.4}$$

Moreover,

$$\varphi_1 - A_0^2\varphi_1 = \tilde{\varphi} + N\int_0^1 \tilde{G}(x,t)\sum_{i=0}^{n-1}\tilde{\varphi}^{(i)}(t)\,dt,$$

from which, as $\tilde{\varphi} \geqq 0$ (in the sense of the ordering of $\tilde{\mathcal{M}}$), we obtain for $0 \leqq x \leqq 1$, $r=0, \ldots, n-1$ that

$$\varphi_1^{(r)}(x) - (A_0^2\varphi_1)^{(r)} \geqq \tilde{\varphi}^{(r)}(x) - \left\{N \max_{0\leqq x\leqq 1}\int_0^1 \left|\frac{\partial^r \tilde{G}(x,t)}{\partial x^r}\right| dt\right\} \max_{0\leqq\xi\leqq 1}\sum_{i=0}^{n-1}\tilde{\varphi}^{(i)}(\xi).$$

By taking a fine enough division of [0, 1] from this we obtain that

$$\varphi_1^{(r)}(x) - (A_0^2\varphi_1)^{(r)} \geqq \delta\lambda_r(1-\theta_2) \quad (0 < \theta_2 < 1). \tag{5.5}$$

This, together with Theorem 2 of [2], implies that

$$0 \leqq A_0^2\varphi_1 \leqq \varphi_1; \quad \varphi_1 \geqq (1-\theta_3)\tilde{\varphi} \quad (\theta_2 < \theta_3 < 1). \tag{5.6}$$

Here θ_2 and θ_3 are absolute constants for all fine enough divisions of the segment [0, 1]. The lemma is proved.

Theorem 5.1. *For problem* (0.1) *under condition* (0.4) *there exist* $Z_1, W_1 \in \tilde{\mathcal{M}}$ *for which the inequalities*

$$Z_2 = E(Z_1, W_1) \leqq Z_1, \quad W_1 \leqq W_2 = F(Z_1, W_1), \tag{5.7}$$

$$Z_3 = E(Z_2, W_2) \leqq Z_1, \quad W_1 \leqq W_3 = F(Z_2, W_2) \tag{5.8}$$

are satisfied in $\tilde{\mathcal{M}}$.

Proof. We are going to seek Z_1, W_1 in the form

$$Z_1 = z_1 + \varphi_1, \quad W_1 = w_1 + \eta_1$$

(cf. Lemma 4.1). Given an arbitrary positive number $\varepsilon > 0$, let us choose $z_1, w_1 \in \tilde{\mathcal{M}}$ so close to the solution y of problem (0.1) as to satisfy the inequalities

$$d_z = \max_{i,x} |z_1^{(i)}(x) - z_2^{(i)}(x)| < \varepsilon, \quad d_w = \max_{i,x} |w_1^{(i)}(x) - w_2^{(i)}(x)| < \varepsilon$$

with $z_2 = E(z_1, w_1)$, $w_2 = F(z_1, w_1)$. The maxima are taken over $i = 0, \ldots, n-1$ and $0 \leqq x \leqq 1$.

Let us now take the solution φ_1 of problem (5.1) with such a $\tilde{\varphi}$ where $\delta = \sqrt{\varepsilon}$. Then for small enough $\varepsilon > 0$ we have, in virtue of (5.2), that

$$\varphi_1 \geqq c\tilde{\varphi} \geqq \varepsilon \geqq d_z, d_w,$$

i.e., with this φ_1 and with $\eta_1 = -\varphi_1$ on account of Lemma 4.1 we obtain that (5.7) is satisfied, i.e.,

$$\varphi^* = Z_1 - Z_2 \geqq 0, \quad \eta^* = W_2 - W_1 \geqq 0.$$

Owing to the formulas of Lemma 4.1 for $\eta_1 = -\varphi_1$ we obtain that

$$(5.9)\quad \begin{aligned} \varphi^*(x) &= z_1(x) - z_2(x) + \tilde{\varphi}(x) - \frac{1}{2}\int_0^1 G(x,t)\left[\sum_{i=0}^{n-1} \varphi_1^{(i)}(t)\left(\frac{\partial f}{\partial u_i}\Big|_P - \frac{\partial f}{\partial u_i}\Big|_Q\right)\right] dt,\\ \eta^*(x) &= w_2(x) - w_1(x) + \tilde{\varphi}(x) + \frac{1}{2}\int_0^1 G(x,t)\left[\sum_{i=0}^{n-1} \varphi_1^{(i)}(t)\left(\frac{\partial f}{\partial u_i}\Big|_P - \frac{\partial f}{\partial u_i}\Big|_Q\right)\right] dt, \end{aligned}$$

where the partial derivatives of f are taken from the formulas

$$f[z_1] - f[z_2] = \sum_{i=0}^{n-1} \frac{\partial f}{\partial u_i}\Big|_P \left(z_1^{(i)}(x) - z_2^{(i)}(x)\right),$$

$$f[w_1] - f[w_2] = \sum_{i=0}^{n-1} \frac{\partial f}{\partial u_i}\Big|_Q \left(w_1^{(i)}(x) - w_2^{(i)}(x)\right).$$

Consequently, as $\varepsilon \to 0$ we get that

$$h(\varepsilon) = \max_{\substack{i=0,\dots,n-1\\ 0 \leqq x \leqq 1}} \left| \frac{\partial f}{\partial u_i}\Big|_{P(x)} - \frac{\partial f}{\partial u_i}\Big|_{Q(x)} \right| \to 0.$$

Expressing (5.8) by $\varphi^* = Z_1 - Z_2, -\eta^* = W_1 - W_2$ as well as (1.4) by $\varphi = z_1 - z_2, -\varphi = w_1 - w_2$ in Theorem 1.1, we get inequalities analogous to (1.6). The latter inequalities are surely satisfied if

$$(5.10)\quad \begin{cases} 1 \geqq \dfrac{-N}{2}\displaystyle\int_0^1 \frac{\partial^r \tilde{G}(x,t)}{\partial x^r} \sum_{i=0}^{n-1} \left(\frac{\max \varphi^{*(i)}}{\min \varphi^{*(r)}} + \frac{\max \eta^{*(i)}}{\min \varphi^{*(r)}}\right) dt + \\ + \dfrac{1}{2}\displaystyle\int_0^1 \left|\frac{\partial^r G(x,t)}{\partial x^r}\right| \sum_{i=0}^{n-1} \max_{0 \leqq \xi \leqq 1} \left| -\frac{\partial f}{\partial u_i}\Big|_P \frac{\varphi^{*(i)}(\xi)}{\min \varphi^{*(r)}} + \frac{\partial f}{\partial u_i}\Big|_Q \frac{\eta^{*(i)}(\xi)}{\min \varphi^{*(r)}} \right| dt \\ (r = 0, \dots, n-1;\ 0 \leqq x \leqq 1) \end{cases}$$

and the same inequality with the rôles of φ^* and η^* interchanged, are satisfied.

From (5.2) it is easy to derive that

$$\frac{1}{2}\int_0^1 \left|\frac{\partial^r \tilde{G}(x,t)}{\partial x^r}\right| \sum_{i=0}^{n-1} \varphi_1^{(i)}(t)\, dt \leqq \frac{1-c}{2N} \tilde{\varphi}^{(r)}(x) \quad (r = 0, \dots, n-1;\ 0 \leqq x \leqq 1).$$

Consequently, as

$$\max_{0 \leqq x \leqq 1} \tilde{\varphi}^{(i)}(x) \leqq \sqrt{\varepsilon}\left(\lambda_i + k(\varepsilon)\right) \quad \left(i = 0, \dots, n-1;\ k(\varepsilon) \to 0 (\varepsilon \to 0)\right)$$

we obtain that

$$\max_{0 \leqq t \leqq 1} \varphi^{*(i)}(t) \leqq \varepsilon + \sqrt{\varepsilon}\left(\lambda_i + k(\varepsilon)\right) + \frac{1-c}{2N}\sqrt{\varepsilon}\left(\lambda_i + k(\varepsilon)\right) h(\varepsilon) =$$

$$= \sqrt{\varepsilon}\left(\lambda_i + a(\varepsilon)\right).$$

Analogously,

$$\min_{0\leqq t\leqq 1} \varphi^{*(i)}(t) \geqq \sqrt{\varepsilon}\,(\lambda_i+b(\varepsilon)), \quad \max_{0\leqq t\leqq 1} \eta^{*(i)}(t) \leqq \sqrt{\varepsilon}\,(\lambda_i+c(\varepsilon)),$$

$$\min_{0\leqq t\leqq 1} \eta^{*(i)}(t) \geqq \sqrt{\varepsilon}(\lambda_i+d(\varepsilon)),$$

where

$$a(\varepsilon), b(\varepsilon), c(\varepsilon), d(\varepsilon) \to 0 \quad (\varepsilon \to 0).$$

From this it follows that the ratios in (5.10) tend to $\dfrac{\lambda_i}{\lambda_r}$ as $\varepsilon\to 0$. Hence the second integral under (5.10) tends to zero and the sum under the first integral to the number

$$\sum_{i=0}^{n-1} \frac{2\lambda_i}{\lambda_r} = \frac{2}{\lambda_r}.$$

Thus for $0\leqq x\leqq 1$ the right hand side of (5.10) tends to a limit not greater than $\theta_1<1$, as $\varepsilon\to 0$. Consequently, for small $\varepsilon>0$ the inequalities (5.8) are also satisfied. This finishes the proof of our theorem.

We remark that Theorem 4.1 guarantees the existence of the required pair $Z_1, W_1\in\mathcal{M}$ (cf. [2]) for the initial value problem

$$\text{(5.11)} \qquad \begin{cases} y^{(n)}(x) = f[y] \equiv f(x, y(x), \dots, y^{(n-1)}(x)) & (0 \leqq x \leqq 1;\ n \geqq 1), \\ y(0) = \dots = y^{(n-1)}(0) = 0 \end{cases}$$

with a function f having the same properties as in problem (0.1) and under the assumption of contractivity (0.4). The general initial value problem on the segment $[a, b]$ can be reduced to problem (5.11).

Analogously as above one can prove the existence of the first pair of approximants for the solution of a boundary value problem that belongs to a wide class of partial differential equations or equations with delayed argument (cf., for example, the references part of [1]).

References

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